Numerical Methods
Using MATLAB
Third Edition

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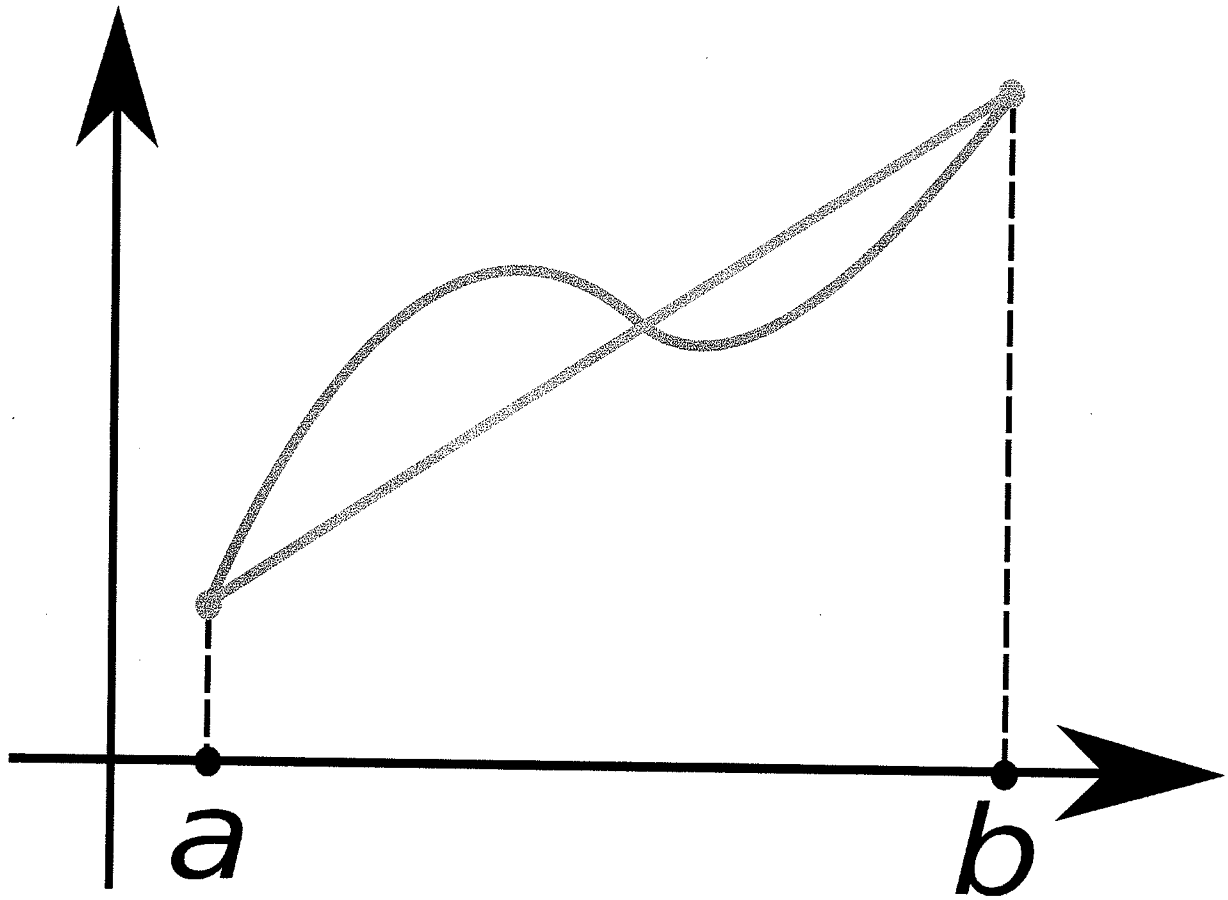
a
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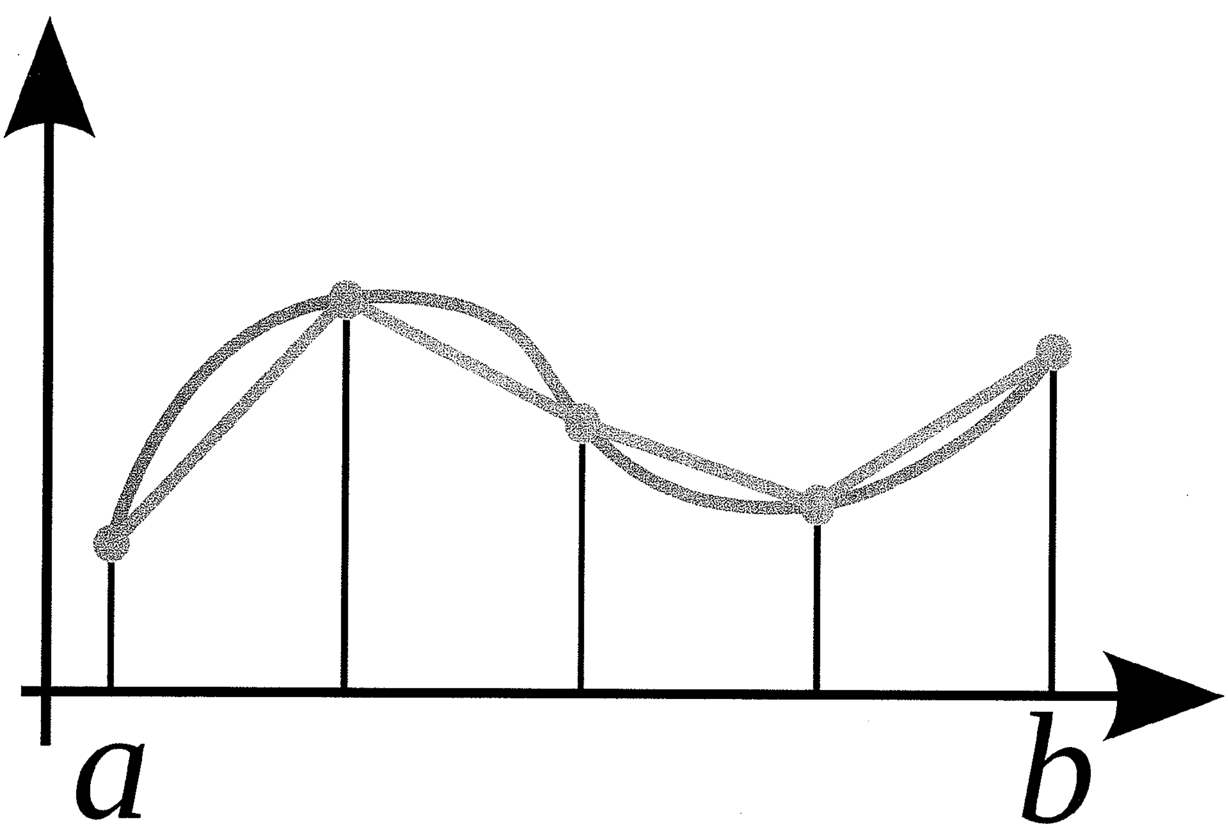
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Theorem 7.2 (Composite Trapezoidal Rule). Suppose that the interval $[a, b]$ is subdivided into M subintervals $[x_k, x_{k+1}]$ of width $h = (b-a)/M$ by using the equally spaced nodes $x_k = a + kh$, for $k = 0, 1, \ldots, M$. The ***composite trapezoidal rule for M subintervals*** can be expressed in any of three equivalent ways:

(1a)
$$T(f, h) = \frac{h}{2}\sum_{k=1}^{M}(f(x_{k-1}) + f(x_k))$$

or

(1b)
$$T(f, h) = \frac{h}{2}(f_0 + 2f_1 + 2f_2 + 2f_3 + \cdots + 2f_{M-2} + 2f_{M-1} + f_M)$$

or

(1c)
$$T(f, h) = \frac{h}{2}(f(a) + f(b)) + h\sum_{k=1}^{M-1} f(x_k).$$

This is an approximation to the integral of $f(x)$ over $[a, b]$, and we write

(2)
$$\int_a^b f(x)\,dx \approx T(f, h).$$

Proof. Apply the trapezoidal rule over each subinterval $[x_{k-1}, x_k]$ (see Figure 7.6). Use the additive property of the integral for subintervals:

(3)
$$\int_a^b f(x)\,dx = \sum_{k=1}^{M}\int_{x_{k-1}}^{x_k} f(x)\,dx \approx \sum_{k=1}^{M}\frac{h}{2}(f(x_{k-1}) + f(x_k)).$$

Since $h/2$ is a constant, the distributive law of addition can be applied to obtain (1a). Formula (1b) is the expanded version of (1a). Formula (1c) shows how to group all the intermediate terms in (1b) that are multiplied by 2. •

Corollary 7.2 (Trapezoidal Rule: Error Analysis). Suppose that $[a, b]$ is subdivided into M subintervals $[x_k, x_{k+1}]$ of width $h = (b - a)/M$. The composite trapezoidal rule

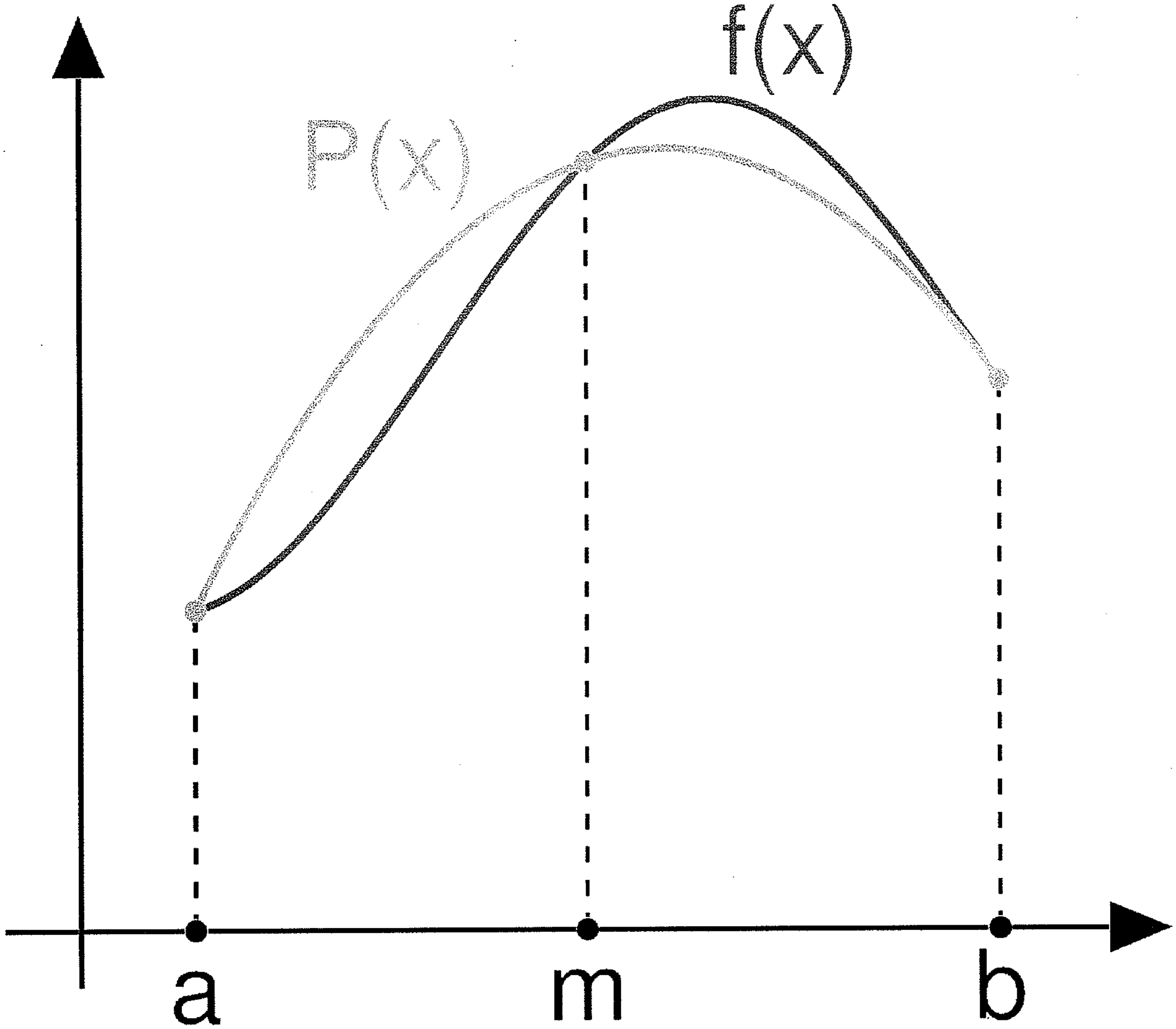
(7)
$$T(f, h) = \frac{h}{2}(f(a) + f(b)) + h\sum_{k=1}^{M-1} f(x_k)$$

is an approximation to the integral

(8)
$$\int_a^b f(x)\,dx = T(f, h) + E_T(f, h).$$

Furthermore, if $f \in C^2[a, b]$, there exists a value c with $a < c < b$ so that the error term $E_T(f, h)$ has the form

(9)
$$E_T(f, h) = \frac{-(b - a)f^{(2)}(c)h^2}{12} = O(h^2).$$


f(x)
P(x)
a
m
b

Theorem 7.3 (Composite Simpson Rule). Suppose that $[a, b]$ is subdivided into $2M$ subintervals $[x_k, x_{k+1}]$ of equal width $h = (b-a)/(2M)$ by using $x_k = a+kh$ for $k = 0, 1, \ldots, 2M$. The ***composite Simpson rule for 2M subintervals*** can be expressed in any of three equivalent ways:

(4a) $$S(f, h) = \frac{h}{3}\sum_{k=1}^{M}(f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k}))$$

or

(4b) $$S(f, h) = \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 2f_{2M-2} + 4f_{2M-1} + f_{2M})$$

or

(4c) $$S(f, h) = \frac{h}{3}(f(a) + f(b)) + \frac{2h}{3}\sum_{k=1}^{M-1} f(x_{2k}) + \frac{4h}{3}\sum_{k=1}^{M} f(x_{2k-1}).$$

This is an approximation to the integral of $f(x)$ over $[a, b]$, and we write

(5) $$\int_a^b f(x)\,dx \approx S(f, h).$$

Proof. Apply Simpson's rule over each subinterval $[x_{2k-2}, x_{2k}]$ (see Figure 7.7). Use the additive property of the integral for subintervals:

(6) $$\begin{aligned}\int_a^b f(x)\,dx &= \sum_{k=1}^{M}\int_{x_{2k-2}}^{x_{2k}} f(x)\,dx \\ &\approx \sum_{k=1}^{M}\frac{h}{3}(f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})).\end{aligned}$$

Since $h/3$ is a constant, the distributive law of addition can be applied to obtain (4a). Formula (4b) is the expanded version of (4a). Formula (4c) groups all the intermediate terms in (4b) that are multiplied by 2 and those that are multiplied by 4. •

Corollary 7.3 (Simpson's Rule: Error Analysis). Suppose that $[a, b]$ is subdivided into $2M$ subintervals $[x_k, x_{k+1}]$ of equal width $h = (b-a)/(2M)$. The composite Simpson rule

(14) $$S(f, h) = \frac{h}{3}(f(a) + f(b)) + \frac{2h}{3}\sum_{k=1}^{M-1} f(x_{2k}) + \frac{4h}{3}\sum_{k=1}^{M} f(x_{2k-1})$$

is an approximation to the integral

(15) $$\int_a^b f(x)\,dx = S(f, h) + E_S(f, h).$$

Furthermore, if $f \in C^4[a, b]$, there exists a value c with $a < c < b$ so that the error term $E_S(f, h)$ has the form

(16) $$E_S(f, h) = \frac{-(b-a)f^{(4)}(c)h^4}{180} = O(h^4).$$

Composite Simpson's Rule

Table 7.3 The Composite ~~Trapezoidal~~ Rule for $f(x) = 2 + \sin(2\sqrt{x})$ over $[1, 6]$

M	h	$S(f, h)$	$E_S(f, h) = O(h^4)$
5	0.5	8.18301549	0.00046371
10	0.25	8.18344750	0.00003171
20	0.125	8.18347717	0.00000204
40	0.0625	8.18347908	0.00000013
80	0.03125	8.18347920	0.00000001

Composite Trapezoidal Rule

Table 7.2 The Composite Trapezoidal Rule for $f(x) = 2 + \sin(2\sqrt{x})$ over $[1, 6]$

M	h	$T(f, h)$	$E_T(f, h) = O(h^2)$
10	0.5	8.19385457	−0.01037540
20	0.25	8.18604926	−0.00257006
40	0.125	8.18412019	−0.00064098
80	0.0625	8.18363936	−0.00016015
160	0.03125	8.18351924	−0.00004003